RELATIONS, FUNCTIONS & INVERSE TRIGONOMETRIC FUNCTIONS

RELATIONS

ORDERED PAIR :

A pair of objects listed in a specific order is called an ordered pair. It is written by listing the two objects in specific order separating them by a comma and then enclosing the pair in parentheses.

In the ordered pair (a, b), a is called the first element and b is called the second element.

Two ordered pairs are set to be equal if their corresponding elements are equal.

i.e. (a, b) = (c, d) if a = c and b = d.

CARTESIAN PRODUCT :

The set of all possible ordered pairs (a, b), where $a \in A$ and $b \in B$ i.e. $\{(a, b) ; a \in A \text{ and } b \in B\}$ is called the Cartesian product of A to B and is denoted by A × B. Usually $A \times B \neq B \times A$.

Similarly $A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$ is called ordered triplet.

RELATION :

Let A and B be two sets. Then a relation R from A to B is a subset of A × B. Thus, R is a relation from A to B $\Rightarrow R \subset A \times B$. The subsets is derived by describing a relationship between the first element and the second element of ordered pairs in A × B e.g. if A = {1, 2, 3, 4, 5, 6, 7, 8} and B = {1, 2, 3, 4, 5} and $R = \{(a, b) : a = b^2, a \in A, b \in B\}$ then R = {(1, 1), (4, 2), (9, 3)}. Here a R b $\Rightarrow$ 1 R 1, 4 R 2, 9 R 3.

NOTE :

(i) Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then A × B consists of mn ordered pairs. So total number of subsets of A × B i.e. number of possible relations from A to B is 2^{mn}.

(ii) A relation R from A to A is called a relation on A.

DOMAIN AND RANGE OF A RELATION :

Let R be a relation from a set A to a set B. Then the set of all first components of coordinates of the ordered pairs belonging to R is called to domain of R, while the set of all second components of coordinates of the ordered pairs in R is called the range of R.

Thus, Dom (R) = $\{a : (a, b) \in R\}$ and Range (R) = $\{b : (a, b) \in R\}$

It is evident from the definition that the domain of a relation from A to B is a subset of A and its range is a subset of B.

Example # 1 : If A = {1, 2} and B = {3, 4}, then find A × B.

Solution : A × B = {(1, 3), (1, 4), (2, 3), (2, 4)}

Example # 2 : Let A = {1, 3, 5, 7} and B = {2, 4, 6, 8} be two sets and let R be a relation from A to B defined by the phrase "$(x, y) \in R \Rightarrow x > y$". Find relation R and its domain and range.

Solution : Under relation R, we have 3R2, 5R2, 5R4, 7R2, 7R4 and 7R6

i.e. R = {(3, 2), (5, 2), (5, 4), (7, 2), (7, 4), (7, 6)}

$\therefore$ Dom (R) = {3, 5, 7} and range (R) = {2, 4, 6}

Example # 3 : Let A = {2, 3, 4, 5, 6, 7, 8, 9}. Let R be the relation on A defined by

$\{(x, y) : x \in A, y \in A \;\&\; x^2 = y \text{ or } x = y^2\}$. Find domain and range of R.

Solution : The relation R is

R = {(2, 4), (3, 9), (4, 2), (9, 3)}

Domain of R = {2, 3, 4, 9}

Range of R = {2, 3, 4, 9}

Self Practice Problem :

(1) If (2x + y, 7) = (5, y – 3) then find x and y.

(2) If A × B = {(1, 2), (1, 3), (1, 6), (7, 2), (7, 3), (7, 6)} then find sets A and B.

(3) If A = {x, y, z} and B = {1, 2} then find number of relations from A to B.

(4) Write R = {(4x + 3, 1 – x) : x ≤ 2, x ∈ N}

Answers (1) $x = -\frac{5}{2}$, y = 10 (2) A = {1, 7}, B = {2, 3, 6}

(3) 64 (4) {(7, 0), (11, –1)}

TYPES OF RELATIONS :

In this section we intend to define various types of relations on a given set A.

(i) **Void relation :** Let A be a set. Then $\phi \subseteq A \times A$ and so it is a relation on A. This relation is called the void or empty relation on A.

(ii) **Universal relation :** Let A be a set. Then $A \times A \subseteq A \times A$ and so it is a relation on A. This relation is called the universal relation on A.

(iii) **Identity relation :** Let A be a set. Then the relation I_A = {(a, a) : a ∈ A} on A is called the identity relation on A. In other words, a relation I_A on A is called the identity relation if every element of A is related to itself only.

(iv) **Reflexive relation :** A relation R on a set A is said to be reflexive if every element of A is related to itself. Thus, R on a set A is not reflexive if there exists an element a ∈ A such that (a, a) ∉ R.

Note : Every identity relation is reflexive but every reflexive relation in not identity.

(v) **Symmetric relation :** A relation R on a set A is said to be a symmetric relation

iff (a, b) ∈ R ⇒ (b ,a) ∈ R for all a, b ∈ A. i.e. a R b ⇒ b R a for all a, b ∈ A.

(vi) **Transitive relation :** Let A be any set. A relation R on A is said to be a transitive relation

iff (a, b) ∈ R and (b, c) ∈ R ⇒ (a, c) ∈ R for all a, b, c ∈ A

i.e. a R b and b R c ⇒ a R c for all a, b, c ∈ A

(vii) **Equivalence relation :** A relation R on a set A is said to be an equivalence relation on A iff

(i) it is reflexive i.e. (a, a) ∈ R for all a ∈ A

(ii) it is symmetric i.e. (a, b) ∈ R ⇒ (b, a) ∈ R for all a, b ∈ A

(iii) it is transitive i.e. (a, b) ∈ R and (b, c) ∈ R ⇒ (a, c) ∈ R for all a, b ∈ A

Example # 4 : Which of the following are identity relations on set A = {1, 2, 3}.
$R_1 = \{(1, 1), (2, 2)\}$, $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$, $R_3 = \{(1, 1), (2, 2), (3, 3)\}$.

Solution: The relation R_3 is identity relation on set A.
R_1 is not identity relation on set A as $(3, 3) \notin R_1$.
R_2 is not identity relation on set A as $(1, 3) \in R_2$

Example # 5 : Which of the following are reflexive relations on set A = {1, 2, 3}.
$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)\}$, $R_2 = \{(1, 1), (3, 3), (2, 1), (3, 2)\}$..

Solution : R_1 is a reflexive relation on set A.
R_2 is not a reflexive relation on A because $2 \in A$ but $(2, 2) \notin R_2$.

Example # 6 : Prove that on the set N of natural numbers, the relation R defined by x R y ⇒ x is less than y is transitive.

Solution : Because for any $x, y, z \in N$ $x < y$ and $y < z \Rightarrow x < z \Rightarrow$ x R y and y R z ⇒ x R z. so R is transitive.

Example # 7 : Let T be the set of all triangles in a plane with R a relation in T given by $R = \{(T_1, T_2) : T_1$ is congruent to $T_2\}$. Show that R is an equivalence relation.

Solution : Since a relation R in T is said to be an equivalence relation if R is reflexive, symmetric and transitive.
(i) Since every triangle is congruent to itself
∴ R is reflexive
(ii) $(T_1, T_2) \in R \Rightarrow T_1$ is congruent to T_2 ⇒ T_2 is congruent to T_1 ⇒ $(T_2, T_1) \in R$
Hence R is symmetric
(iii) Let $(T_1, T_2) \in R$ and $(T_2, T_3) \in R$ ⇒ T_1 is congruent to T_2 and T_2 is congruent to T_3
⇒ T_1 is congruent to T_3 ⇒ $(T_1, T_3) \in R$
∴ R is transitive
Hence R is an equivalence relation.

Example # 8 : Show that the relation R in R defined as $R = \{(a, b) : a \le b\}$ is transitive.

Solution : Let $(a, b) \in R$ and $(b, c) \in R$
∴ $(a \le b)$ and $b \le c$ ⇒ $a \le c$ ∴ $(a, c) \in R$ Hence R is transitive.

Example # 9 : Show that the relation R in the set {1, 2, 3} given by R = {(1, 2), (2, 1)} is symmetric.

Solution : Let $(a, b) \in R$ [∵ $(1, 2) \in R$]
∴ $(b, a) \in R$ [∵ $(2, 1) \in R$]
Hence R is symmetric.

Self Practice Problem :

(5) Let L be the set of all lines in a plane and let R be a relation defined on L by the rule $(x, y) \in R$ ⇒ x is perpendicular to y. Then prove that R is a symmetric relation on L.

(6) Let R be a relation on the set of all lines in a plane defined by $(\ell_1, \ell_2) \in R \Rightarrow$ line ℓ_1 is parallel to line ℓ_2. Prove that R is an equivalence relation.

FUNCTION

Definition :

Function is a rule (or correspondence), from a non empty set A to a non empty set B, that associates each member of A to a unique member of B. Symbolically, we write f: A $\rightarrow$ B. We read it as "f is a function from A to B".

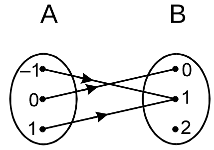
For example, let A $\equiv$ {–1, 0, 1} and B $\equiv$ {0, 1, 2}.

Then A × B $\equiv$ {(–1, 0), (–1, 1), (–1, 2), (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)}

Now, " f : A $\rightarrow$ B defined by f(x) = x^2 " is the function such that

f $\equiv$ {(–1, 1), (0, 0), (1, 1)}

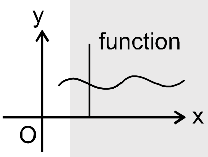
f can also be shown diagramatically by following mapping.



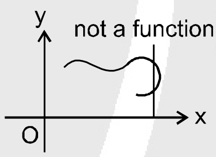
Note : Every function say y = f(x) : A $\rightarrow$ B. Here x is independent variable which takes its values from A while 'y' takes its value from B. A relation will be a function if and only if

(i) x must be able to take each and every value of A and

(ii) one value of x must be related to one and only one value of y in set B.



(a)



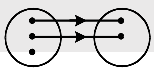
(b)

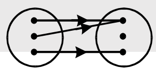
Graphically : If any vertical line cuts the graph at more than one point, then the graph does not represent a function.

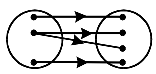
Example # 10 : (i) Which of the following correspondences can be called a function ?

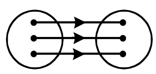
(A)	f(x) = x^3	;	{–1, 0, 1} $\rightarrow$ {0, 1, 2, 3}
(B)	f(x) = $\pm\sqrt{x}$	;	{0, 1, 4} $\rightarrow$ {–2, –1, 0, 1, 2}
(C)	f(x) = $\sqrt{x}$	;	{0, 1, 4} $\rightarrow$ {–2, –1, 0, 1, 2}
(D)	f(x) = $-\sqrt{x}$	;	{0, 1, 4} $\rightarrow$ {–2, –1, 0, 1, 2}

(ii) Which of the following pictorial diagrams represent the function

(A) 

(B) 

(C) 

(D) 

Solution :

(i) f(x) in (C) and (D) are functions as definition of function is satisfied. while in case of (A) the given relation is not a function, as f(–1) $\notin$ 2nd set. Hence definition of function is not satisfied. While in case of (B), the given relation is not a function, as f(1) = ± 1 and f(4) = ± 2 i.e. element 1 as well as 4 in 1st set is related with two elements of 2nd set.Hence definition of function is not satisfied.

(ii) B and D. In (A) one element of domain has no image, while in (C) one element of 1st set has two images in 2nd set

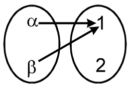
Self practice problem :

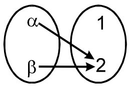
(7) Let g(x) be a function defined on [–1, 1]. If the area of the equilateral triangle with two of its vertices at (0,0) and (x,g(x)) is $\sqrt{3}/4$ sq. unit, then the function g(x) may be.

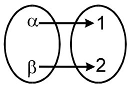
(A) g(x) = $\pm\sqrt{(1-x^2)}$ (B) g(x) = $\sqrt{(1-x^2)}$ (C) g(x) = $-\sqrt{(1-x^2)}$ (D) g(x) = $\sqrt{(1+x^2)}$

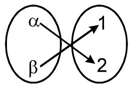
(8) Represent all possible functions defined from {α, β} to {1, 2}.

Answers : (7) B, C

(8) (i) 

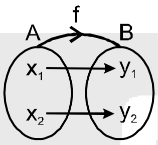
(ii) 

(iii) 

(iv) 

Domain, Co-domain and Range of a Function :

Let y = f(x) : A → B, then the set A is known as the domain of f and the set B is known as co-domain of f.



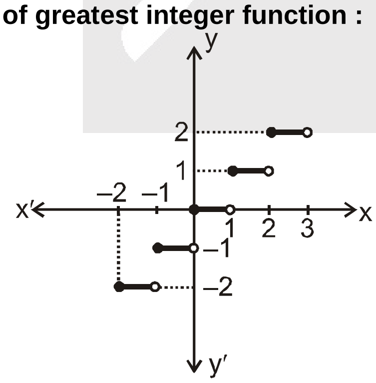
If x_1 is mapped to y_1, then y_1 is called as image of x_1 under f. Further x_1 is a pre-image of y_1 under f.
If only expression of f (x) is given (domain and co-domain are not mentioned), then domain is **complete** set of those values of x for which f (x) is real, while codomain is considered to be (– ∞, ∞) (except in inverse trigonometric functions).
Range is the complete set of values that y takes. Clearly range is a subset of Co-domain.
A function whose domain and range are both subsets of real numbers is called **a real function**.

Greatest Integer Function / Fractional Part Function / Signum Function / Drichlet's Function :

(i) Greatest integer function or step up function :

The function y = f (x) = [x] is called the greatest integer function where [x] equals to the greatest integer less than or equal to x. For example :
[3.2] = 3; [– 3.2] = – 4
for $-1 \le x < 0$; [x] = – 1 ; for $0 \le x < 1$; [x] = 0
for $1 \le x < 2$; [x] = 1 ; for $2 \le x < 3$; [x] = 2 and so on.

Graph of greatest integer function :



Properties of greatest integer function :

(a) $x - 1 < [x] \le x$

(b) $[x \pm m] = [x] \pm m$ iff m is an integer.

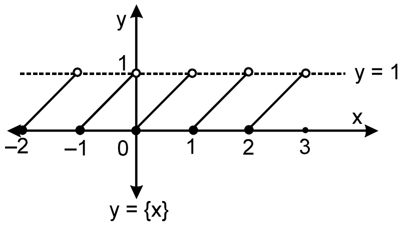
(c) $[x] + [y] \le [x + y] \le [x] + [y] + 1$

(d) $[x] + [-x] = \begin{cases} 0; & \text{if } x \text{ is an integer} \\ -1 & \text{otherwise} \end{cases}$

Note : $[mx] \ne m[x]$

(ii) Fractional part function:

It is defined as y = {x} = x – [x]. It is always non-negative and varies from [0, 1). The period of this function is 1 and graph of this function is as shown.



For example $\{2.1\} = 2.1 - [2.1] = 2.1 - 2 = 0.1$
$\{-3.7\} = -3.7 - [-3.7] = -3.7 + 4 = 0.3$

Properties of fractional part function :

(a) $\{x \pm m\} = \{x\}$ iff m is an integer

(b) $\{x\} + \{-x\} = \begin{cases} 0 \ , & \text{if } x \text{ is an integer} \\ 1 \ , & \text{otherwise} \end{cases}$

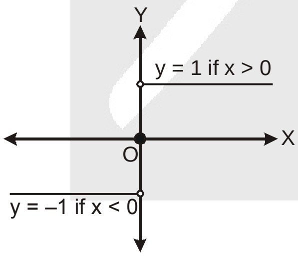
Note: $\{mx\} \neq m\{x\}$

(iii) Signum function :

A function f (x) = sgn (x) is defined as follows :

$$f(x) = \text{sgn}(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$

It is also written as $\text{sgn}(x) = \begin{cases} \dfrac{|x|}{x} \ ; & x \neq 0 \\ 0 \ ; & x = 0 \end{cases}$



y = sgn x

Note : ☞ $\text{sgn}(f(x)) = \begin{cases} \dfrac{|f(x)|}{f(x)} \ ; & f(x) \neq 0 \\ 0 \ ; & f(x) = 0 \end{cases}$

(iv) Dirichlet Function :

Let c and d ≠ c be real numbers (usually taken as c = 1 and d = 0). The Dirichlet function is defined by

$$D(x) = \begin{cases} c & \text{for } x \text{ rational} \\ d & \text{for } x \text{ irrational} \end{cases}$$

Example # 11 : Solve the equation [x] + {–x} = 2x, (where [.] and { . } represents greatest integer function and fractional part function respectively).

Solution : **case-I** $x \in I$

$x + 0 = 2x \Rightarrow x = 0$

case-II $x \notin I$

$[x] + 1 - \{x\} = 2x$

$[I + f] + 1 - \{I + f\} = 2(I + f)$

$I + 1 - f = 2I + 2f$

$\frac{1-I}{3} = f$ as $0 < f < 1$

$0 < \frac{1-I}{3} < 1$

$0 < 1 - I < 3$

$-1 < -I < 2$

$-2 < I < 1 \Rightarrow I = -1, 0$

$f = \frac{2}{3}, \frac{1}{3}$

Here $x = -\frac{1}{3}, \frac{1}{3}$ $\therefore$ Solutions are $x = 0, -\frac{1}{3}, \frac{1}{3}$

Algebraic Operations on Functions :

If f and g are real valued functions of x with domain set A and B respectively, then both f and g are defined in $A \cap B$. Now we define f + g, f – g, (f . g) and (f /g) as follows:

(i) $(f \pm g)(x) = f(x) \pm g(x)$

(ii) $(f.g)(x) = f(x).\, g(x)$

— domain in each case is $A \cap B$

(iii) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ domain is $\{x \mid x \in A \cap B$ such that $g(x) \neq 0\}$.

Note : ☞ For domain of $\phi(x) = \{f(x)\}^{g(x)}$, conventionally, the conditions are f(x) > 0 and g(x) must be real.

☞ For domain of $\phi(x) = {}^{f(x)}C_{g(x)}$ or $\phi(x) = {}^{f(x)}P_{g(x)}$ conventional conditions of domain are $f(x) \geq g(x)$ and $f(x) \in N$ and $g(x) \in W$.

Example # 12 : Find the domain of following functions :

(i) $f(x) = \sqrt{x^2 - 5}$ (ii) $\sin(x^3 - x)$

Solution : (i) $f(x) = \sqrt{x^2 - 5}$ is real iff $x^2 - 5 \geq 0$

$\Rightarrow |x| \geq \sqrt{5} \Rightarrow x \leq -\sqrt{5}$ or $x \geq \sqrt{5}$

$\therefore$ the domain of f is $(-\infty, -\sqrt{5}] \cup [\sqrt{5}, \infty)$

(ii) $x^3 - x \in R$ $\therefore$ domain is $x \in R$

Example # 13 : Find the domain of function $f(x) = \frac{3}{\sqrt{4-x^2}} \log(x^3 - x)$

Solution : Domain of $\sqrt{4-x^2}$ is [–2, 2] but $\sqrt{4-x^2} = 0$ for $x = \pm 2 \Rightarrow x \in (-2, 2)$

$\log(x^3 - x)$ is defined for $x^3 - x > 0$ i.e. $x(x-1)(x+1) > 0$.

$\therefore$ domain of $\log(x^3 - x)$ is $(-1, 0) \cup (1, \infty)$.

Hence the domain of the given function is $\{(-1, 0) \cup (1, \infty)\} \cap (-2, 2) \equiv (-1, 0) \cup (1, 2)$.

Self practice problems :

(9) Find the domain of following functions.

(i) $f(x) = \dfrac{1}{\log(2-x)} + \sqrt{x+1}$ (ii) $f(x) = \sqrt{1-x} - \sin\dfrac{2x-1}{3}$

Answers : (i) $[-1, 1) \cup (1, 2)$ (ii) $[-1, 1]$

Methods of determining range :

(i) **Representing x in terms of y**

If y = f(x), try to express as x = g(y), then domain of g(y) represents possible values of y, which is range of f(x).

(ii) **Graphical Method :**

The set of y– coordinates of the graph of a function is the range.

Example # 14 : Find the range of $f(x) = \dfrac{x^2+x+1}{x^2+x-1}$

Solution : $f(x) = \dfrac{x^2+x+1}{x^2+x-1}$ {$x^2 + x + 1$ and $x^2 + x - 1$ have no common factor}

$y = \dfrac{x^2+x+1}{x^2+x-1}$

$\Rightarrow$ $yx^2 + yx - y = x^2 + x + 1$

$\Rightarrow$ $(y - 1)\,x^2 + (y - 1)\,x - y - 1 = 0$

If y = 1, then the above equation reduces to –2 = 0. Which is not true.

Further if $y \neq 1$, then $(y - 1)\,x^2 + (y - 1)\,x - y - 1 = 0$ is a quadratic and has real roots if

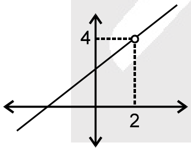
$(y - 1)^2 - 4\,(y - 1)\,(-y - 1) \geq 0$

i.e. if $y \leq -3/5$ or $y \geq 1$ but $y \neq 1$

Thus the range is $(-\infty, -3/5] \cup (1, \infty)$

Example # 15 : Find the range of $f(x) = \dfrac{x^2-4}{x-2}$

Solution :

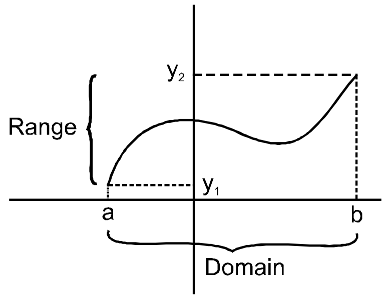


$f(x) = \dfrac{x^2-4}{x-2} = x + 2;\ x \neq 2$

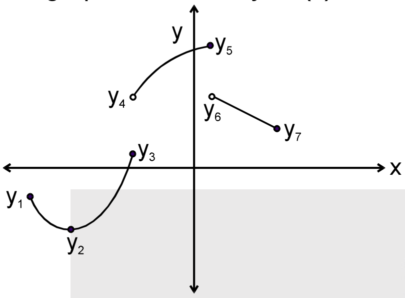
$\therefore$ graph of f(x) would be

Thus the range of f(x) is $R - \{4\}$

Further if f(x) happens to be continuous in its domain then range of f(x) is [min f(x), max. f(x)]. However for sectionally continuous functions, range will be union of [min f(x), max. f(x)] over all those intervals where f(x) is continuous, as shown by following example.



Example # 16 : Let graph of function y = f(x) is



Then range of above sectionally continuous function is $[y_2, y_3] \cup [y_7, y_6) \cup (y_4, y_5]$

(iii) **Using monotonocity :** Many of the functions are monotonic increasing or monotonic decreasing. In case of monotonic continuous functions the minimum and maximum values lie at end points of domain. Some of the common function which are increasing or decreasing in the interval where they are **continuous** is as under.

Monotonic increasing	Monotonic decreasing
$\log_a x$, a > 1	$\log_a x$, 0 < a < 1
e^x	e^{-x}
$\sin^{-1} x$	$\cos^{-1} x$
$\tan^{-1} x$	$\cot^{-1} x$
$\sec^{-1} x$	$\text{cosec}^{-1} x$

For monotonic increasing functions in [a, b]

(i) $f'(x) \geq 0$ (ii) range is [f(a), f(b)]

for monotonic decreasing functions in [a, b]

(i) $f'(x) \leq 0$ (ii) range is [f(b), f(a)]

Example # 17 : Find the range of function $y = \ell n\,(2x - x^2)$

Solution : **Step – 1**

We have $2x - x^2 \in (-\infty, 1]$

Step – 2 Let $t = 2x - x^2$

For ℓnt to be defined accepted values are (0, 1]

Now, using monotonocity of $\ell n\, t$,

$\ell n\,(2x - x^2) \in (-\infty, 0]$

$\therefore$ range is $(-\infty, 0]$ **Ans.**

Self practice problems :

(10) Find domain and range of following functions.

(i) $y = x^3$ (ii) $y = \dfrac{x^2 - 2x + 5}{x^2 + 2x + 5}$ (iii) $y = \dfrac{1}{\sqrt{x^2 - x}}$

Answers : (i) domain R; range R (ii) domain R ; range $\left[\dfrac{3-\sqrt{5}}{2}, \dfrac{3+\sqrt{5}}{2}\right]$

(iii) domain R – [0, 1] ; range $(0, \infty)$

Classification of Functions :

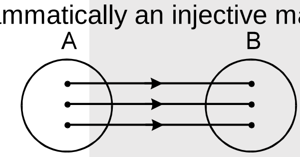
Functions can be classified as "One – One Function (Injective Mapping)" and "Many – One Function" :

One - One Function :

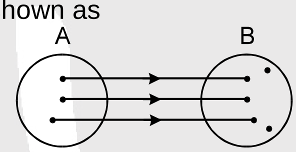
A function $f : A \to B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B.

Thus for $x_1, x_2 \in A$ and $f(x_1), f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Diagrammatically an injective mapping can be shown as



OR

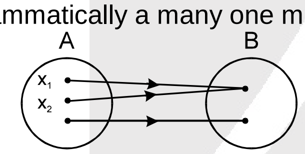


Many - One function :

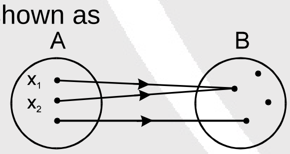
A function $f : A \to B$ is said to be a many one function if there exist at least two or more elements of A having the same f image in B.

Thus $f : A \to B$ is many one iff there exist atleast two elements $x_1, x_2 \in A$, such that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Diagrammatically a many one mapping can be shown as



OR



Note : ☞ If a function is one–one, it cannot be many–one and vice versa.

Methods of determining whether a given function is ONE-ONE or MANY-ONE :

(a) If $x_1, x_2 \in A$ and $f(x_1), f(x_2) \in B$, equate $f(x_1)$ and $f(x_2)$ and if it implies that $x_1 = x_2$, then and only then function is ONE-ONE otherwise MANY-ONE.

(b) If there exists a straight line parallel to x-axis, which cuts the graph of the function atleast at two points, then the function is MANY-ONE, otherwise ONE- ONE.

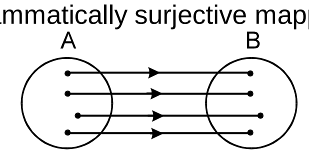
(c) If either $f'(x) \geq 0$, $\forall$ x $\in$ domain or $f'(x) \leq 0$ $\forall$ x $\in$ domain, where equality can hold at discrete point(s) only i.e. strictly monotonic, then function is ONE-ONE, otherwise MANY-ONE.

Note : If f and g both are one-one, then gof and fog would also be one-one (if they exist). Functions can also be classified as "Onto function (Surjective mapping)" and "Into function":

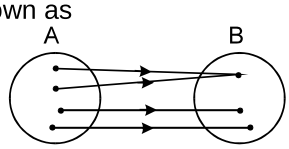
Onto function :

If the function $f : A \to B$ is such that each element in B (co–domain) must have atleast one pre–image in A, then we say that f is a function of A 'onto' B. Thus $f : A \to B$ is surjective iff $\forall$ b $\in$ B, there exists some a $\in$ A such that f (a) = b.

Diagrammatically surjective mapping can be shown as



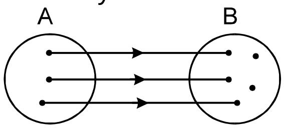
OR



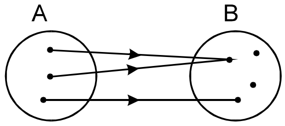
Into function :

If f : A → B is such that there exists atleast one element in co–domain which is not the image of any element in domain, then f(x) is into.

Diagrammatically into function can be shown as



OR

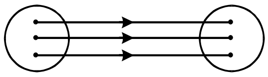


Note : (i) If range ≡ co–domain, then f(x) is onto, otherwise into

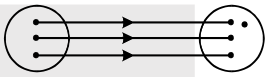
(ii) If a function is onto, it cannot be into and vice versa.

A function can be one of these four types:

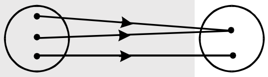
(a) one–one onto (injective and surjective)



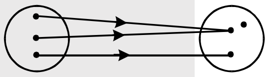
(b) one–one into (injective but not surjective)



(c) many–one onto (surjective but not injective)



(d) many–one into (neither surjective nor injective)



Note :

(i) If f is both injective and surjective, then it is called a **bijective** mapping. The bijective functions are also named as invertible, non singular or biuniform functions.

(ii) If a set A contains 'n' distinct elements, then the number of different functions defined from A → A is n^n and out of which n! are one one.

(iii) If f and g both are onto, then gof or fog may or may not be onto.

(iv) The composite of two bijections is a bijection iff f and g are two bijections such that gof is defined, then gof is also a bijection only **when co-domain of f is equal to the domain of g**.

Example # 18 : (i) Find whether f(x) = x + cos x is one-one.

(ii) Identify whether the function $f(x) = -x^3 + 3x^2 - 2x + 4$ for f : R → R is ONTO or INTO

(iii) $f(x) = x^2 - 2x$; [0, 3] → A. Find whether f(x) is injective or not. Also find the set A, if f(x) is surjective.

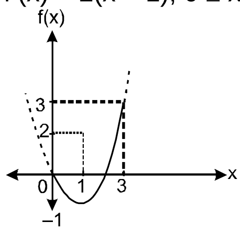
Solution : (i) The domain of f(x) is R. $f'(x) = 1 - \sin x$.

∴ $f'(x) \ge 0\ \forall\ x \in$ complete domain and equality holds at discrete points only

∴ f(x) is strictly increasing on R. Hence f(x) is one-one.

(ii) As range ≡ codomain, therefore given function is ONTO

(iii) $f'(x) = 2(x - 1);\ 0 \le x \le 3$



∴ $f'(x) = \begin{cases} -ve & ; \quad 0 \le x < 1 \\ +ve & ; \quad 1 < x < 3 \end{cases}$

∴ f(x) is non monotonic. Hence it is not injective.

For f(x) to be surjective, A should be equal to its range. By graph range is [–1, 3]

∴ A ≡ [–1, 3]

Self practice problems :

(11) For each of the following functions find whether it is one-one or many-one and also into or onto

(i) $f(x) = 2 \tan x;\ (\pi/2, 3\pi/2) \to R$ (ii) $f(x) = \dfrac{1}{1+x^2}$; $(-\infty, 0) \to R$

(iii) $f(x) = x^2 + \ell n\, x$

Answers : (i) one-one onto (ii) one-one into (iii) one-one onto

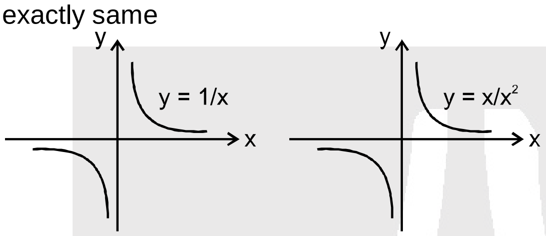
Equal or Identical Functions :

Two functions f and g are said to be identical (or equal) iff :

(i) The domain of f $\equiv$ the domain of g.

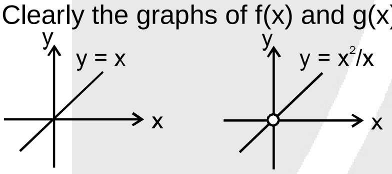
(ii) f(x) = g(x), for every x belonging to their common domain.

e.g. $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{x}{x^2}$ are identical functions. Clearly the graphs of f(x) and g(x) are exactly same



But f(x) = x and $g(x) = \dfrac{x^2}{x}$ are not identical functions.

Clearly the graphs of f(x) and g(x) are different at x = 0.



Example # 19 : Examine whether following pair of functions are identical or not ?

(i) $f(x) = \dfrac{x^2-1}{x-1}$ and $g(x) = x + 1$

(ii) $f(x) = \sin^2x + \cos^2x$ and $g(x) = \sec^2x - \tan^2x$

Solution : (i) No, as domain of f(x) is R – {1}
while domain of g(x) is R

(ii) No, as domain are not same. Domain of f(x) is R
while that of g(x) is $R - \left\{(2n+1)\dfrac{\pi}{2};\ n \in I\right\}$

Self practice problems

(12) Examine whether the following pair of functions are identical or not :

(i) f(x) = sgn (x) and $g(x) = \begin{cases} \dfrac{x}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$

(ii) $f(x) = \operatorname{cosec}^2x - \cot^2x$ and g(x) = 1

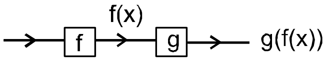
Answers : (i) Yes (ii) No

Composite Function :

Let f: $X \to Y_1$ and g: $Y_2 \to Z$ be two functions and D is the set of values of x such that if $x \in X$, then $f(x) \in Y_2$. If $D \neq \phi$, then the function h defined on D by h(x) = g{f(x)} is called composite function of g and f and is denoted by gof. It is also called function of a function.

Note : ☞ Domain of gof is D which is a subset of X (the domain of f). Range of gof is a subset of the range of g. If D = X, then $f(X) \subseteq Y_2$.

Pictorially gof(x) can be viewed as under



Note that gof(x) exists only for those x when range of f(x) is a subset of domain of g(x).

Properties of Composite Functions :

(a) In general gof ≠ fog (i.e. not commutative)

(b) The composition of functions are associative i.e. if three functions f, g, h are such that fo (goh) and (fog) oh are defined, then fo (goh) = (fog) oh.

Example # 20 : Describe fog and gof wherever is possible for the following functions

(i) $f(x) = \sqrt{x+3}$, $g(x) = 1 + x^2$ (ii) $f(x) = \sqrt{x}$, $g(x) = x^2 - 1$.

Solution : (i) Domain of f is [–3, ∞), range of f is [0, ∞).

Domain of g is R, range of g is [1, ∞).

For gof(x)

Since range of f is a subset of domain of g,

∴ domain of gof is [–3, ∞) {equal to the domain of f }

$gof(x) = g\{f(x)\} = g(\sqrt{x+3}) = 1 + (x+3) = x + 4$. Range of gof is [1, ∞).

For fog(x)

since range of g is a subset of domain of f,

∴ domain of fog is R {equal to the domain of g}

$fog(x) = f\{g(x)\} = f(1+x^2) = \sqrt{x^2+4}$ Range of fog is [2, ∞).

(ii) $f(x) = \sqrt{x}$, $g(x) = x^2 - 1$.

Domain of f is [0, ∞), range of f is [0, ∞).

Domain of g is R, range of g is [–1, ∞).

For gof(x)

Since range of f is a subset of the domain of g,

∴ domain of gof is [0, ∞) and $g\{f(x)\} = g(\sqrt{x}) = x - 1$. Range of gof is [–1, ∞)

For fog(x)

Since range of g is not a subset of the domain of f

i.e. $[-1, \infty) \not\subset [0, \infty)$

∴ fog is not defined on whole of the domain of g.

Domain of fog is {x∈R, the domain of g : g(x)∈ [0, ∞), the domain of f}.

Thus the domain of fog is $D = \{x \in R: 0 \le g(x) < \infty\}$

i.e. $D = \{ x \in R: 0 \le x^2 - 1\} = \{ x \in R: x \le -1 \text{ or } x \ge 1 \} = (-\infty, -1] \cup [1, \infty)$

$fog(x) = f\{g(x)\} = f(x^2-1) = \sqrt{x^2-1}$ Its range is [0, ∞).

Example # 21 : Let $f(x) = e^x$; $R^+ \to R$ and $g(x) = \sin x$; $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$. Find domain and range of fog(x)

Solution : Domain of f(x) : (0, ∞) Range of g(x) : [–1, 1]

values in range of g(x) which are accepted by f(x) are $\left(0, \frac{\pi}{2}\right]$

$\Rightarrow \quad 0 < g(x) \le 1 \quad \Rightarrow \quad 0 < \sin x \le 1 \quad \Rightarrow \quad 0 < x \le \frac{\pi}{2}$

Hence domain of fog(x) is $x \in (0, \frac{\pi}{2}]$

Therefore Domain : $(0, \frac{\pi}{2}]$

Range : (1, e]

Example # 22 : If $f(x) = -1 + |x - 2|,\ 0 \le x \le 4$

$g(x) = 2 - |x|,\ -1 \le x \le 3$

Then find fog (x) and gof (x). Also draw their rough sketch.

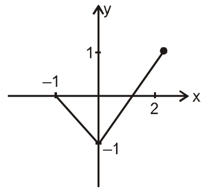
Solution : $\text{fog}(x) = \{-1 + |g(x) - 2|,\ 0 \le g(x) \le 4,\ -1 \le x \le 3$

$= \{-1 + |2 - |x| - 2|,\ 0 \le 2 - |x| \le 4,\ -1 \le x \le 3$

$= \{-1 + |x|,\ -2 \le x \le 2,\ -1 \le x \le 3$

$$= \begin{cases} -(1+x), & -1 \le x \le 0 \\ x - 1, & 0 < x \le 2 \end{cases}\ ;$$

graph of fog(x) is



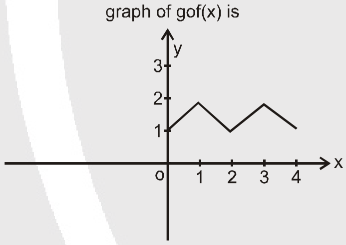
$\text{gof}(x) = \{2 - |f(x)|,\ -1 \le f(x) \le 3,\ 0 \le x \le 4$

$= \{2 - |-1 + |x - 2||,\ -1 \le -1 + |x - 2| \le 3,\ 0 \le x \le 4$

$= \{2 - |-1 + |x - 2||,\ -2 \le x \le 6,\ 0 \le x \le 4$

$$= \begin{cases} x + 1, & 0 \le x < 1 \\ 3 - x, & 1 \le x \le 2 \\ x - 1, & 2 < x \le 3 \\ 5 - x, & 3 < x \le 4 \end{cases}\ ;$$

graph of gof(x) is



Self practice problems

(13) Define fog(x) and gof(x). Also find their domain and range.

(i) $f(x) = [x],\ g(x) = \sin x$

(ii) $f(x) = \tan x,\ x \in (-\pi/2, \pi/2);\ g(x) = \sqrt{1 - x^2}$

(14) Let $f(x) = e^x : R^+ \to R$ and $g(x) = x^2 - x : R \to R$. Find domain and range of fog (x) and gof (x)

Answers :

(13) (i) gof = sin [x] domain : R range { sin a : a ∈ I}

fog = [sin x] domain : R range : {–1, 0, 1}

(ii) $\text{gof} \equiv \sqrt{1 - \tan^2 x}$, domain : $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ range : [0, 1]

$\text{fog} \equiv \tan\sqrt{1 - x^2}$ domain : [–1, 1] range [0, tan 1]

(14) fog (x) | gof (x)

Domain : $(-\infty, 0) \cup (1, \infty)$ | Domain : $(0, \infty)$

Range : $(1, \infty)$ | Range : $(0, \infty)$

Odd and Even Functions :

(i) If f (–x) = f (x) for all x in the domain of 'f', then f is said to be an even function.
e.g. f (x) = cos x; g (x) = $x^2 + 3$.

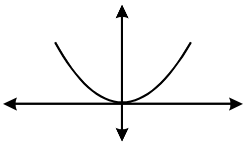
(ii) If f (–x) = –f (x) for all x in the domain of 'f', then f is said to be an odd function.
e.g. f (x) = sin x; g (x) = $x^3 + x$.

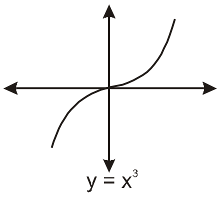
Note : (i) A function may neither be odd nor even. (e.g. $f(x) = e^x$, $\cos^{-1}x$)

(ii) If an odd function is defined at x = 0, then f(0) = 0

Properties of Even/Odd Function

(a) The graph of every even function is symmetric about the y–axis and that of every odd function is symmetric about the origin.
For example graph of $y = x^2$ is symmetric about y-axis, while graph of $y = x^3$ is symmetric about origin


$y = x^2$


$y = x^3$

(b) All functions (whose domain is symmetrical about origin) can be expressed as the sum of an even and an odd function, as follows

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd}}$$

(c) The only function which is defined on the entire number line and is even and odd at the same time is $f(x) = 0$.

(d) If f and g both are even or both are odd, then the function f.g will be even but if any one of them is odd and the other even then f.g will be odd.

(e) If f(x) is even then $f'(x)$ is odd while derivative of odd function is even. Note that same cannot be said for integral of functions.

Example # 23 : Show that $a^x + a^{-x}$ is an even function.
Solution : Let $f(x) = a^x + a^{-x}$
Then $f(-x) = a^{-x} + a^{-(-x)} = a^{-x} + a^x = f(x)$. Hence f(x) is an even function

Example # 24 : Prove that $f(x) = x\left(\frac{x}{e^x-1}+\frac{x}{2}\right)$ is odd function

Solution : Let $g(x) = \left(\frac{x}{e^x-1}+\frac{x}{2}\right)$ then $g(-x) = \left(\frac{-x}{e^{-x}-1}+\frac{-x}{2}\right) = \left(\frac{x}{e^x-1}+\frac{x}{2}\right)$
$\Rightarrow$ g(x) is even
hence $f(x) = x.g(x) = x\left(\frac{x}{e^x-1}+\frac{x}{2}\right)$ is odd function.

Self practice problems

(15) Determine whether the following functions are even / odd / neither even nor odd?

(i) $f(x) = \dfrac{e^x+e^{-x}}{e^x-e^{-x}}$

(ii) $f : [-2, 3] \to [0, 9]$, $f(x) = x^2$

(iii) $f(x) = x \log\left(x+\sqrt{x^2+1}\right)$

Answers (i) Odd (ii) neither even nor odd (iii) Even

Periodic Functions :

A function f(x) is called periodic with a period T if there exists a real number T > 0 such that for each x in the domain of f the numbers x – T and x + T are also in the domain of f and f(x) = f(x + T) for all x in the domain of f(x). Graph of a periodic function with period T is repeated after every interval of 'T'.
e.g. The function sin x and cos x both are periodic over 2π and tan x is periodic over π.
The least positive period is called the principal or fundamental period of f(x) or simply the period of the function.

Note : Inverse of a periodic function does not exist.

Properties of Periodic Functions :

(a) If f(x) has a period T, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also have a period T.

(b) If f(x) has a period T, then f (ax + b) has a period . $\frac{T}{|a|}$

(c) Every constant function defined for all real x, is always periodic, with no fundamental period.

(d) If f (x) has a period T_1 and g (x) also has a period T_2 then period of f(x) ± g(x) or f(x) . g(x) or $\frac{f(x)}{g(x)}$ is L.C.M. of T_1 and T_2 provided their L.C.M. exists. However that L.C.M. (if exists) need not to be fundamental period. If L.C.M. does not exists then f(x) ± g(x) or f(x) . g(x) or $\frac{f(x)}{g(x)}$ is nonperiodic.

L.C.M. of $\left(\frac{a}{b}, \frac{p}{q}, \frac{\ell}{m}\right) = \frac{\text{L.C.M.}(a,\ p,\ \ell)}{\text{H.C.F.}(b,\ q,\ m)}$

e.g. |sinx| has the period π, | cosx | also has the period π

$\therefore$ |sinx| + |cosx| also has a period π. But the fundamental period of |sinx| + |cosx| is $\frac{\pi}{2}$.

(e) If g is a function such that gof is defined on the domain of f and f is periodic with T, then gof is also periodic with T as one of its periods.

Example # 25 : Find period of the following functions

(i) $f(x) = \sin\frac{x}{2} + \cos\frac{x}{3}$

(ii) f(x) = {x} + sin x, where {.}denotes fractional part function

(iii) f(x) = 4 cos x . cos 3x + 2 (iv) $f(x) = \sin\frac{3x}{2} - \cos\frac{x}{3} - \tan\frac{2x}{3}$

Solution : (i) Period of $\sin\frac{x}{2}$ is 4π while period of $\cos\frac{x}{3}$ is 6π . Hence period of $\sin\frac{x}{2} + \cos\frac{x}{3}$ is 12π

{L.C.M. of 4 and 6 is 12}

(ii) Period of sin x $= 2\pi$

Period of {x} = 1

but L.C.M. of 2π and 1 is not possible as their ratio is irrational number

$\therefore$ it is aperiodic

(iii) f(x) = 4 cos x . cos 3x + 2

period of f(x) is L.C.M. of $\left(2\pi, \frac{2\pi}{3}\right) = 2\pi$

but 2π may or may not be fundamental periodic, but fundamental period $= \frac{2\pi}{n}$, where $n \in N$. Hence cross-checking for n = 1, 2, 3, ...we find π to be fundamental period

$f(\pi + x) = 4(-\cos x)(-\cos 3x) + 2 = f(x)$

(iv) Period of f(x) is L.C.M. of $\frac{2\pi}{3/2}, \frac{2\pi}{1/3}, \frac{\pi}{2/3}$ = L.C.M. of $\frac{4\pi}{3}, 6\pi, \frac{3\pi}{2} = 12\pi$

Inverse of a Function :

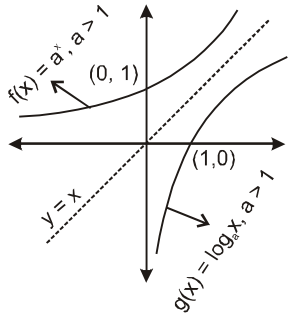
Let y = f(x) : A $\to$ B be a one-one and onto function. i.e. bijection, then there will always exist bijective function x = g(y) : B $\to$ A such that if (p, q) is an element of f, (q, p) will be an element of g and the functions f(x) and g(x) are said to be inverse of each other. g(x) is also denoted by $f^{-1}(x)$ and f(x) is denoted by $g^{-1}(x)$

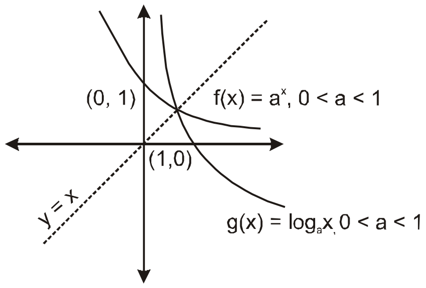
Note : (i) The inverse of a bijection is unique.

(ii) Inverse of an even function is not defined.

Properties of Inverse Function :

(a) The graphs of f and g are the mirror images of each other in the line y = x. For example $f(x) = a^x$ and $g(x) = \log_a x$ are inverse of each other, and their graphs are mirror images of each other on the line y = x as shown below.





(b) Normally points of intersection of f and f^{-1} lie on the straight line y = x. However it must be noted that f(x) and $f^{-1}(x)$ may intersect otherwise also. e.g f(x) = 1/x

(c) In general fog(x) and gof(x) are not equal. But if f and g are inverse of each other, then gof = fog. fog(x) and gof(x) can be equal even if f and g are not inverse of each other. e.g. f(x) = x + 1, g(x) = x + 2. However if fog(x) = gof(x) = x, then $g(x) = f^{-1}(x)$

(d) If f and g are two bijections f : A → B, g : B → C, then the inverse of gof exists and $(gof)^{-1} = f^{-1} \text{ o } g^{-1}$.

(e) If f(x) and g(x) are inverse function of each other, then $f'(g(x)) = \dfrac{1}{g'(x)}$

Example # 26 : (i) Determine whether $f(x) = \dfrac{2x+3}{4}$ for f : R → R, is invertible or not? If so find it.

(ii) Let $f(x) = x^2 + 2x;\ x \ge -1$. Draw graph of $f^{-1}(x)$ also find the number of solutions of the equation, $f(x) = f^{-1}(x)$

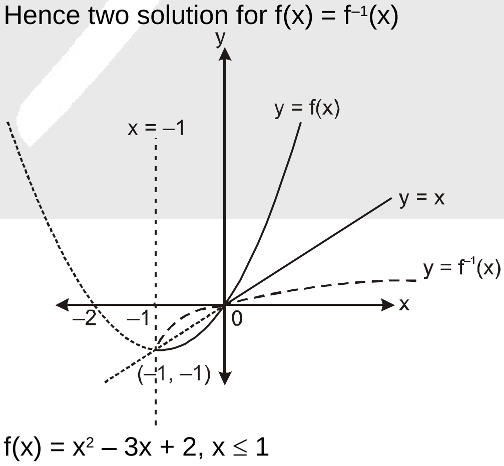
(iii) If $y = f(x) = x^2 - 3x + 2,\ x \le 1$. Find the value of g′(2)where g is inverse of f

Solution : (i) Given function is one-one and onto, therefore it is invertible.

$y = \dfrac{2x+3}{4} \quad \Rightarrow \quad x = \dfrac{4y-3}{2} \quad \therefore \quad f^{-1}(x) = \dfrac{4x-3}{2}$

(ii) $f(x) = f^{-1}(x)$ is equivalent to $f(x) = x \quad \Rightarrow \quad x^2 + 2x = x \Rightarrow x(x + 1) = 0 \Rightarrow x = 0, -1$

Hence two solution for $f(x) = f^{-1}(x)$



(iii) $f(x) = x^2 - 3x + 2,\ x \le 1$

$f(g(x) = g(x)^2 - 3\, g(x) + 2$

$\Rightarrow \quad 2 = g(2)^2 - 3g(2) + 2$

$\Rightarrow \quad g(2) = 0,\ 3 \le 1$

so g(2) = 0

$f'(x) = 2x - 3$

$f(g(x) = x \Rightarrow f'(g(x))\ .\ g'(x) = 1 \Rightarrow g'(2) = \dfrac{1}{f'(g(2))} = \dfrac{1}{f'(0)} = -\dfrac{1}{3}$

Self practice problems :

(16) Determine $f^{-1}(x)$, if given function is invertible

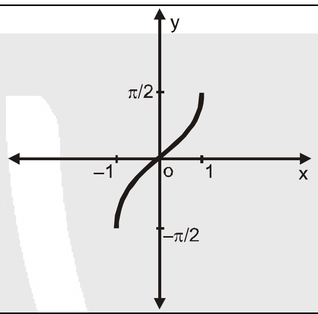
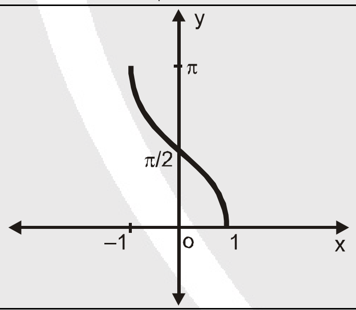
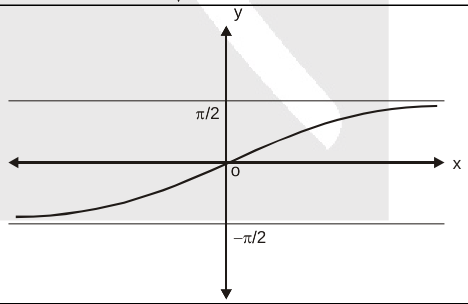
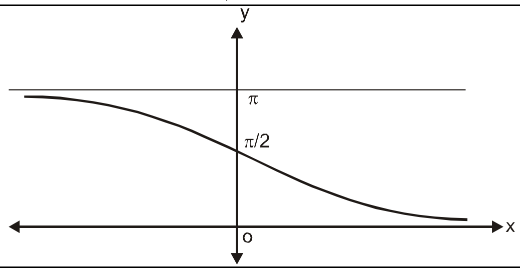
$f : (-\infty, -1) \to (-\infty, -2)$ defined by $f(x) = -(x + 1)^2 - 2$

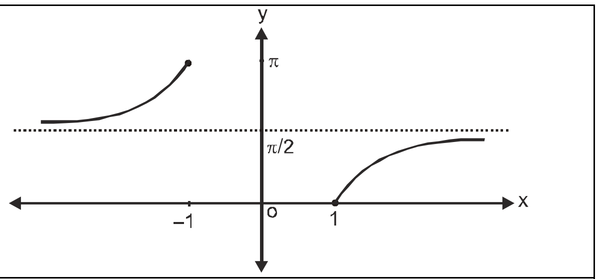
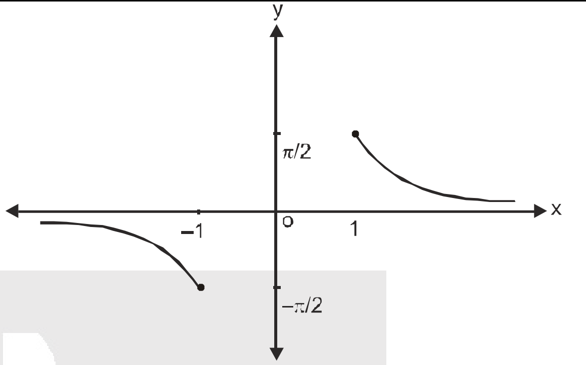
Answers : $-1 - \sqrt{-x-2}$

Inverse Trigonometry Functions

<u>Introduction</u> : The student may be familiar about trigonometric functions viz sin x, cos x, tan x, cosec x, sec x, cot x with respective domains R, R, R – {(2n + 1) π/2}, R – {nπ}, R – {(2n + 1) π/2}, R – {nπ} and respective ranges [–1, 1], [–1, 1], R, R – (–1, 1), R – (–1, 1), R.

Correspondingly, six inverse trigonometric functions (also called inverse circular functions) are defined.

Inverse Trigonometric Function	Domain	Range	Graph
$f(x) = \sin^{-1}x$ or arcsinx	[–1, 1]	[–π/2, π/2]	
$f(x) = \cos^{-1}x$ or arccosx	[–1, 1]	[0, π]	
$f(x) = \tan^{-1}x$ or arctanx	R	(–π/2, π/2)	
$f(x) = \cot^{-1}x$ or arccotx	R	(0, π)	

$f(x) = \sec^{-1}x$ or arcsecx	$R - (-1, 1)$	$[0, \pi] - \{\pi/2\}$	
$f(x) = \text{cosec}^{-1}x$ or arccosecx	$R - (-1, 1)$	$[-\pi/2, \pi/2] - \{0\}$	

Example # 27 : Find the value of $\tan\left[\cos^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$

Solution : $\tan\left[\cos^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right] = \tan\left[\frac{\pi}{3} + \left(-\frac{\pi}{6}\right)\right] = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$.

Example # 28 : Find domain of $\sin^{-1}(2x^2 + 1)$

Solution : Let $y = \sin^{-1}(2x^2 + 1)$

For y to be defined $-1 \le (2x^2 + 1) \le 1 \Rightarrow -2 \le 2x^2 \le 0 \Rightarrow x \in \{0\}$

Self practice problems :

(17) Find the value of (i) $\cos\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$

(ii) $\text{cosec}\,[\sec^{-1}(\sqrt{2}) + \cot^{-1}(1)]$

(18) Find the domain of

(i) $y = \sec^{-1}(x^2 + 3x + 1)$

(ii) $y = \sin^{-1}\left(\frac{x^2}{1+x^2}\right)$

(iii) $y = \cot^{-1}(\sqrt{x^2 - 1})$

(19) Find the range of (i) $\sin^{-1}|x| + \sec^{-1}|x|$

(ii) $\sin^{-1}\sqrt{x^2 + x + 1}$

Answers :

(17) (i) 0 (ii) 1

(18) (i) $(-\infty, -3] \cup [-2, -1] \cup [0, \infty)$ (ii) R

(iii) $(-\infty, -1] \cup [1, \infty)$

(19) (i) $\{\pi/2\}$ (ii) $[\pi/3, \pi/2]$

Property 1 : T(T^{-1})

(i) $\sin(\sin^{-1} x) = x, \quad -1 \le x \le 1$

Proof : Let $\theta = \sin^{-1}x$. Then $x \in [-1, 1]$ & $\theta \in [-\pi/2, \pi/2]$.

$\Rightarrow \quad \sin\theta = x$, by meaning of the symbol $\quad \Rightarrow \quad \sin(\sin^{-1} x) = x$

Similar proofs can be carried out to obtain

(ii) $\cos(\cos^{-1} x) = x, \quad -1 \le x \le 1$

(iii) $\tan(\tan^{-1} x) = x, \ x \in R$

(iv) $\cot(\cot^{-1} x) = x, \ x \in R$

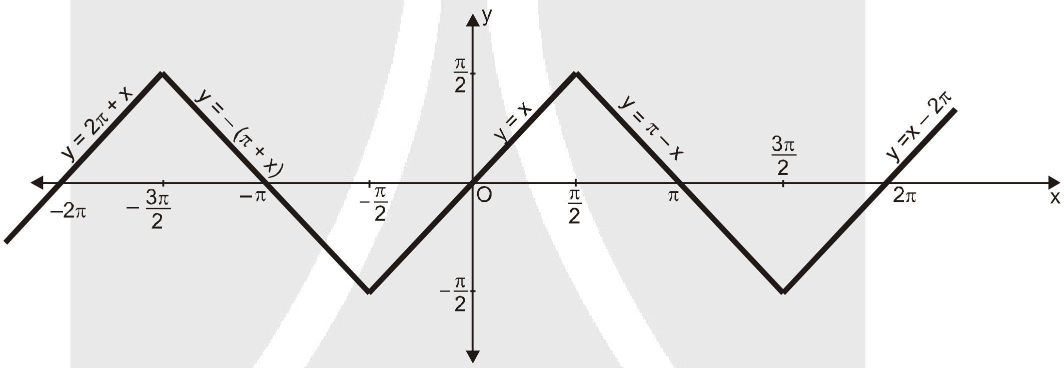
(v) $\sec(\sec^{-1} x) = x, \quad x \le -1, x \ge 1$

(vi) $\text{cosec}(\text{cosec}^{-1} x) = x, \quad |x| \ge 1$

Property 2 : T^{-1}(T)

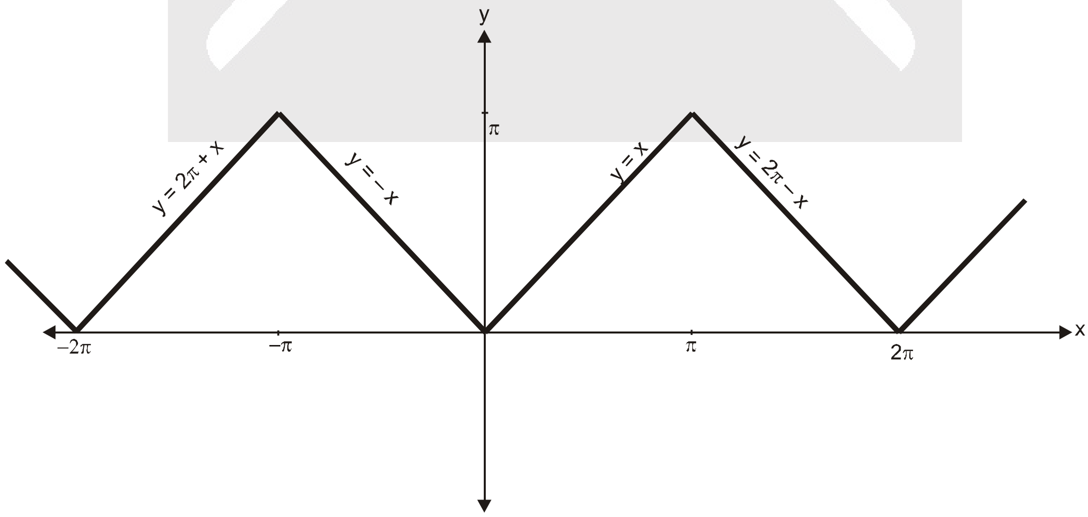
(i) $\sin^{-1}(\sin x) = \begin{cases} -2n\pi + x, & x \in [2n\pi - \pi/2,\ 2n\pi + \pi/2] \\ (2n+1)\ \pi - x, & x \in [(2n+1)\ \pi - \pi/2,\ (2n+1)\pi + \pi/2],\ n \in Z \end{cases}$

Graph of $y = \sin^{-1}(\sin x)$



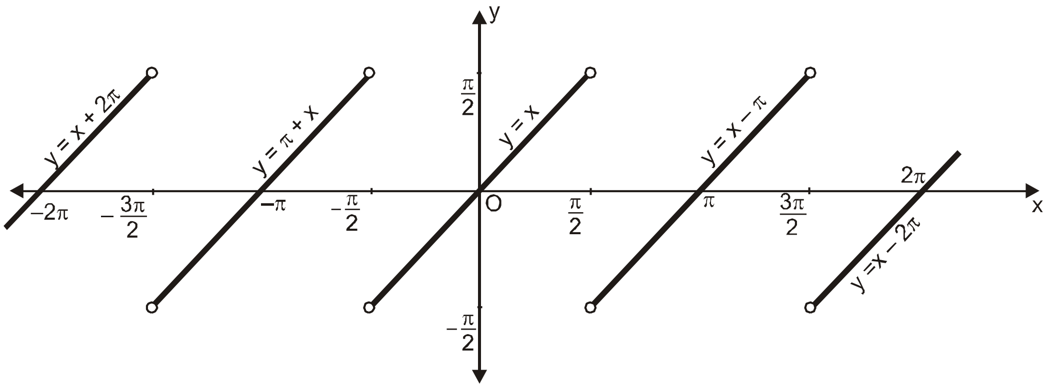
(ii) $\cos^{-1}(\cos x) = \begin{cases} -2n\pi + x, & x \in [2n\pi,\ (2n+1)\pi] \\ 2n\pi - x, & x \in [(2n-1)\ \pi,\ 2n\pi],\ n \in I \end{cases}$

Graph of $y = \cos^{-1}(\cos x)$



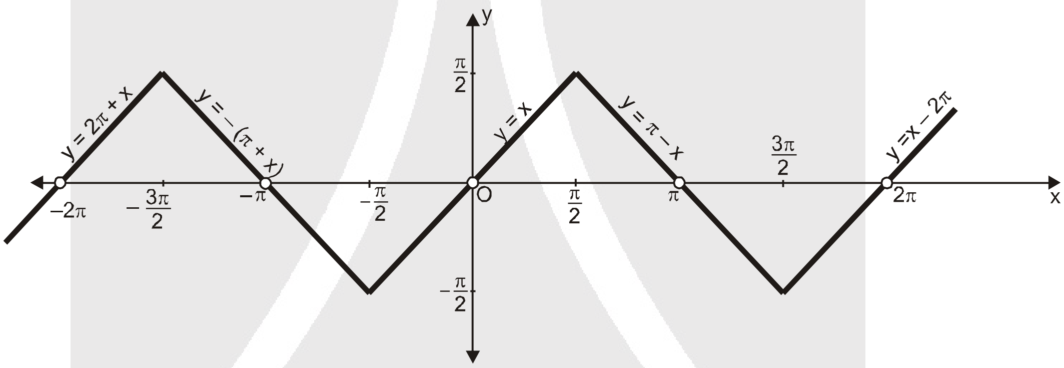
(iii) $\tan^{-1}(\tan x) = -n\pi + x,\ n\pi - \pi/2 < x < n\pi + \pi/2,\ n \in z$

Graph of $y = \tan^{-1}(\tan x)$



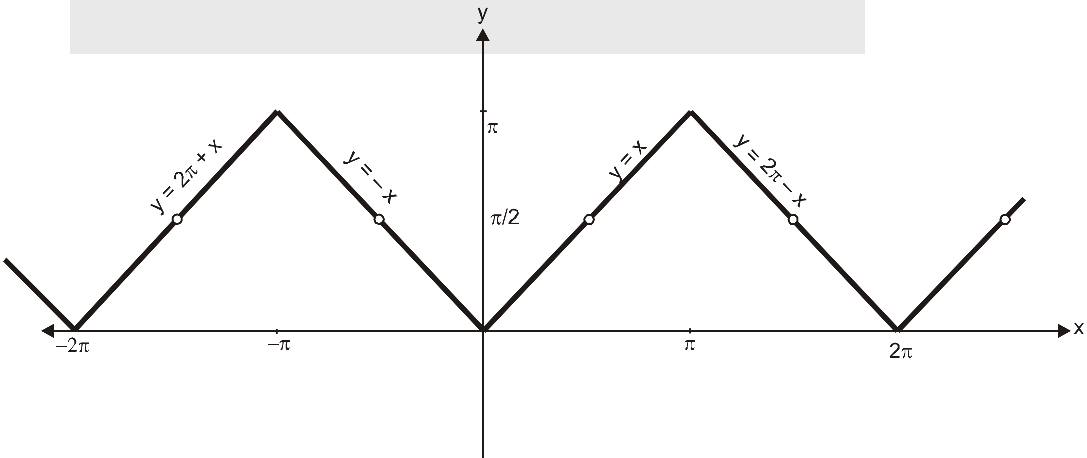
(iv) $\text{cosec}^{-1}(\text{cosec } x)$ is similar to $\sin^{-1}(\sin x)$

Graph of $y = \text{cosec}^{-1}(\text{cosec } x)$



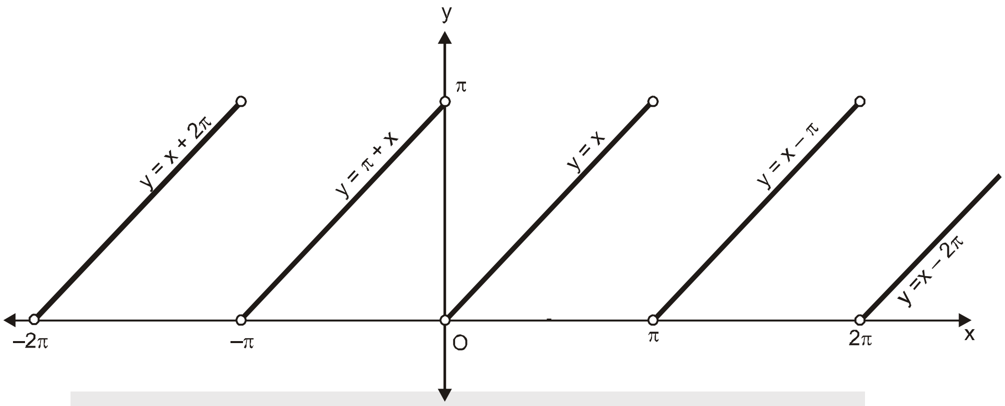
(v) $\sec^{-1}(\sec x)$ is similar to $\cos^{-1}(\cos x)$

Graph of $y = \sec^{-1}(\sec x)$



(vii) $\cot^{-1}(\cot x) = -n\pi + x,\ x \in (n\pi, (n+1)\pi),\ n \in \mathbb{Z}$

Graph of $y = \cot^{-1}(\cot x)$



Remark : $\sin(\sin^{-1}x)$, $\cos(\cos^{-1}x)$, ... $\cot(\cot^{-1}x)$ are aperiodic (non periodic) functions where as $\sin^{-1}(\sin x)$, ..., $\cot^{-1}(\cot x)$ are periodic functions.

Property 3 : "–x"

The graphs of $\sin^{-1}x$, $\tan^{-1}x$, $\text{cosec}^{-1}x$ are symmetric about origin.

Hence we get $\sin^{-1}(-x) = -\sin^{-1}x$

$\tan^{-1}(-x) = -\tan^{-1}x$

$\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$.

Also the graphs of $\cos^{-1}x$, $\sec^{-1}x$, $\cot^{-1}x$ are symmetric about the point $(0, \pi/2)$. From this, we get

$\cos^{-1}(-x) = \pi - \cos^{-1}x$

$\sec^{-1}(-x) = \pi - \sec^{-1}x$

$\cot^{-1}(-x) = \pi - \cot^{-1}x$.

Property 4 : "π/2"

(i) $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2},\ -1 \le x \le 1$

Proof : Let $A = \sin^{-1}x$ and $B = \cos^{-1}x$ $\Rightarrow$ $\sin A = x$ and $\cos B = x$

$\Rightarrow$ $\sin A = \cos B$ $\Rightarrow$ $\sin A = \sin(\pi/2 - B)$

$\Rightarrow$ $A = \pi/2 - B$, because A and $\pi/2 - B \in [-\pi/2, \pi/2]$

$\Rightarrow$ $A + B = \pi/2$.

Similarly, we can prove

(ii) $\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2},\ x \in R$ (iii) $\text{cosec}^{-1}x + \sec^{-1}x = \dfrac{\pi}{2},\ |x| \ge 1$

Example # 29 : Find the value of cosec $\left\{\cot\left(\cot^{-1}\frac{3\pi}{4}\right)\right\}$.

Solution :

$\because$ $\cot(\cot^{-1} x) = x, \ \forall\ x \in R$

$\therefore$ $\cot\left(\cot^{-1}\frac{3\pi}{4}\right) = \frac{3\pi}{4}$

$\text{cosec}\left\{\cot\left(\cot^{-1}\frac{3\pi}{4}\right)\right\} = \text{cosec}\left(\frac{3\pi}{4}\right) = .\ \sqrt{2}$

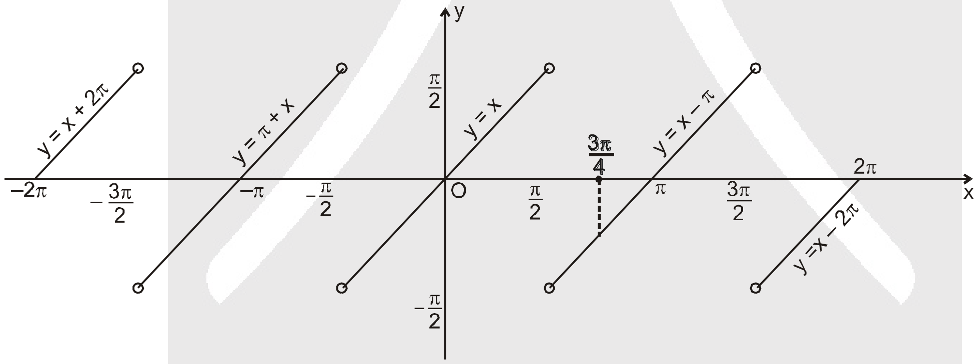
Example # 30 Find the value of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$.

Solution : $\because$ $\tan^{-1}(\tan x) = x$ if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

As $\frac{3\pi}{4} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\therefore$ $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) \neq \frac{3\pi}{4}$

$\therefore$ $\frac{3\pi}{4} \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

graph of $y = \tan^{-1}(\tan x)$ is as :



$\because$ from the graph we can see that if $\frac{\pi}{2} < x < \frac{3\pi}{2}$,

then $\tan^{-1}(\tan x) = x - \pi$

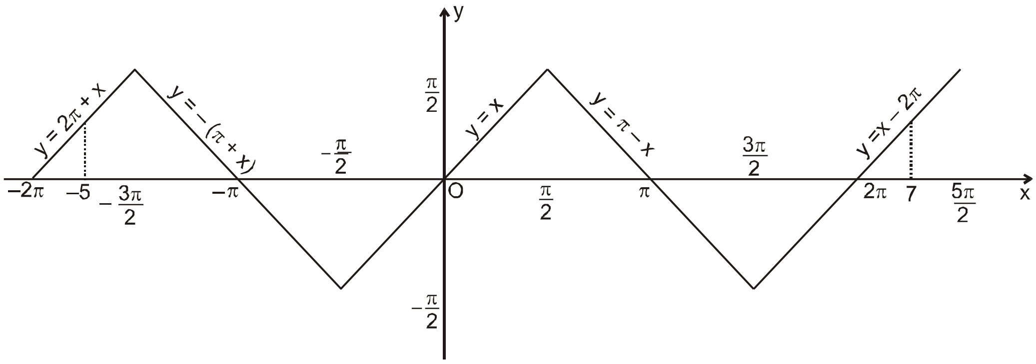
$\therefore$ $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \frac{3\pi}{4} - \pi = -\frac{\pi}{4}$

Example # 31 : Find the value of $\sin^{-1}(\sin 7)$ and $\sin^{-1}(\sin(-5))$.

Solution. Let $y = \sin^{-1}(\sin 7)$

$\sin^{-1}(\sin 7) \neq 7$ as $7 \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\because$ $2\pi < 7 < \frac{5\pi}{2}$

graph of $y = \sin^{-1}(\sin x)$ is as :



From the graph we can see that if $2\pi \le x \le \frac{5\pi}{2}$, then

$y = \sin^{-1}(\sin x)$ can be written as :

$y = x - 2\pi$

$\therefore \quad \sin^{-1}(\sin 7) = 7 - 2\pi$

Similarly if we have to find $\sin^{-1}(\sin(-5))$ then

$\because \quad -2\pi < -5 < -\frac{3\pi}{2}$

$\therefore$ from the graph of $\sin^{-1}(\sin x)$, we can say that $\sin^{-1}(\sin(-5)) = 2\pi + (-5) = 2\pi - 5$

Example # 32 : Solve $\sin^{-1}(x^2 - 2x + 1) + \cos^{-1}(x^2 - x) = \frac{\pi}{2}$

Solution : $\sin^{-1}(f(x)) + \cos^{-1}(g(x)) = \frac{\pi}{2} \Leftrightarrow f(x) = g(x)$ and $-1 \le f(x), g(x) \le 1$

$x^2 - 2x + 1 = x^2 - x \Leftrightarrow x = 1$, accepted as a solution

Self practice problems :

(20) Find the value of (i) $\cos\left\{\sin\left(\sin^{-1}\frac{\pi}{6}\right)\right\}$

(ii) $\sin\left\{\cos\left(\cos^{-1}\frac{3\pi}{4}\right)\right\}$ (iii) $\cos^{-1}(\cos 13)$

(iv) $\cos^{-1}(-\cos 4)$ (v) $\tan^{-1}\left\{\tan\left(-\frac{7\pi}{8}\right)\right\}$ (vi) $\tan^{-1}\left\{\cot\left(-\frac{1}{4}\right)\right\}$

(21) Find $\sin^{-1}(\sin\theta)$, $\cos^{-1}(\cos\theta)$, $\tan^{-1}(\tan\theta)$, $\cot^{-1}(\cot\theta)$ for $\theta \in \left(\frac{5\pi}{2}, 3\pi\right)$

(22) Solve the following equations (i) $5\tan^{-1}x + 3\cot^{-1}x = 2\pi$ (ii) $4\sin^{-1}x = \pi - \cos^{-1}x$

(iii) Solve $\sin^{-1}(x^2 - 2x + 3) + \cos^{-1}(x^2 - x) = \frac{\pi}{2}$

Answer : (20) (i) $\frac{\sqrt{3}}{2}$ (ii) not defined (iii) $13 - 4\pi$

(iv) $4 - \pi$ (v) $\frac{\pi}{8}$ (vi) $\left(\frac{1}{4} - \frac{\pi}{2}\right)$

(21) $3\pi - \theta$, $\theta - 2\pi$, $\theta - 3\pi$, $\theta - 2\pi$

(22). (i) $x = 1$ (ii) $x = \frac{1}{2}$ (iii) No solution

Interconversion & Simplification

Interconversion of any trigonometric ratio inverse means its conversion in remaining five trigonometric ratio inverse. Example

$$\sin^{-1}x = \begin{cases} \text{for } x \in (0,1) & \text{for } x \in (-1,0) \\ \cos^{-1}\sqrt{1-x^2} & -\cos^{-1}\sqrt{1-x^2} \\ \tan^{-1}\dfrac{x}{\sqrt{1-x^2}} & \tan^{-1}\dfrac{x}{\sqrt{1-x^2}} \\ \cot^{-1}\dfrac{\sqrt{1-x^2}}{x} & -\pi+\cot^{-1}\dfrac{\sqrt{1-x^2}}{x} \\ \sec^{-1}\dfrac{1}{\sqrt{1-x^2}} & -\sec^{-1}\dfrac{1}{\sqrt{1-x^2}} \\ \operatorname{cosec}^{-1}\dfrac{1}{x} & \operatorname{cosec}^{-1}\dfrac{1}{x} \end{cases}$$

Example # 33 : Convert (i) $\tan^{-1}3$, (ii) $\sin^{-1}(-1/3)$ in terms of cosine inverse.

Sol. (i) Let $\theta = \tan^{-1}3 \Rightarrow \tan\theta = 3 \Rightarrow \cos\theta = \dfrac{1}{\sqrt{10}} \Rightarrow \theta = \cos^{-1}\dfrac{1}{\sqrt{10}}$

(ii) $\sin^{-1}(-1/3) = -\sin^{-1}(1/3)$

Let $\theta = \sin^{-1}(1/3) \Rightarrow \sin\theta = \dfrac{1}{3} \Rightarrow \cos\theta = \dfrac{2\sqrt{2}}{3} \Rightarrow \theta = \cos^{-1}\dfrac{2\sqrt{2}}{3}$

$\Rightarrow \quad \sin^{-1}(-1/3) = -\cos^{-1}\dfrac{2\sqrt{2}}{3}$

Example # 34 : Show that $\cot^{-1}x = \begin{cases} \tan^{-1}(1/x), & x > 0 \\ \pi + \tan^{-1}(1/x), & x < 0 \end{cases}$

Sol. Let $\cot^{-1}x = \theta$ ($x = \cot\theta$) $\Rightarrow \quad \theta \in (0, \pi)$

Now $\tan^{-1}(1/x) = \tan^{-1}\tan(\theta) \quad = \begin{cases} \theta & \theta \in \left(0, \dfrac{\pi}{2}\right) \\ \theta - \pi & \theta \in \left(\dfrac{\pi}{2}, \pi\right) \end{cases}$

$$= \begin{cases} \cot^{-1}x & x > 0 \\ \cot^{-1}x - \pi & x < 0 \end{cases}$$

$\Rightarrow \quad \cot^{-1}x = \begin{cases} \tan^{-1}(1/x), & x > 0 \\ \pi + \tan^{-1}(1/x), & x < 0 \end{cases}$

Example # 35 : Show that $\sin^{-1}\frac{2x}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| \le 1 \\ \pi - 2\tan^{-1}x & \text{if } x > 1 \\ -(\pi + 2\tan^{-1}x) & \text{if } x < -1 \end{cases}$

Sol : Let $\tan^{-1} x = \theta (x = \tan\theta) \Rightarrow \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow 2\theta \in (-\pi, \pi)$

Now $\sin^{-1}\frac{2x}{1+x^2} = \sin^{-1}\sin 2\theta = \begin{cases} -\pi - 2\theta & 2\theta \in \left(-\pi, -\frac{\pi}{2}\right] \text{ or } \theta \in \left(-\frac{\pi}{2}, -\frac{\pi}{4}\right] \\ 2\theta & 2\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ or } \theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \\ \pi - 2\theta & 2\theta \in \left[\frac{\pi}{2}, \pi\right) \text{ or } \theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right) \end{cases}$

$$= \begin{cases} 2\tan^{-1}x & \text{if } x \in [-1, 1] \\ \pi - 2\tan^{-1}x & \text{if } x \ge 1 \\ -(\pi + 2\tan^{-1}x) & \text{if } x \le -1 \end{cases}$$

Example # 36 : Define $y = \cos^{-1}(4x^3 - 3x)$ in terms of $\cos^{-1} x$

Solution : Let $\cos^{-1}x = \theta (x = \cos\theta) \Rightarrow \theta \in [0, \pi] \Rightarrow 3\theta = [0, 3\pi]$

Now $y = \cos^{-1}(4x^3 - 3x) = \cos^{-1}\cos 3\theta$

$$= \begin{cases} 3\theta & 3\theta \in [0, \pi] \text{ or } \theta \in \left[0, \frac{\pi}{3}\right] \\ 2\pi - 3\theta & 3\theta \in [\pi, 2\pi] \text{ or } \theta \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right] \\ 3\theta - \pi & 3\theta \in [2\pi, 3\pi] \text{ or } \theta \in \left[\frac{2\pi}{3}, \pi\right] \end{cases}$$

$$y = \cos^{-1}(4x^3 - 3x) = \begin{cases} 3\cos^{-1}x & ; \frac{1}{2} \le x \le 1 \\ 2\pi - 3\cos^{-1}x & ; -\frac{1}{2} \le x < \frac{1}{2} \\ -2\pi + 3\cos^{-1}x & ; -1 \le x < -\frac{1}{2} \end{cases}$$

Example # 37 : Simplify (i) $\sin\left(\pi \tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}\right)$

(ii) $\sin\left(2\tan^{-1}\frac{1}{2}\right)$

(iii) $\cos(2\cos^{-1}(1/5) + \sin^{-1}(1/5))$

Solution : (i) Let $y = \tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}$ ....(A)

$\because$ $\cot^{-1}(-x) = \pi - \cot^{-1}x,\ x \in R$

(A) can be written as

$y = \tan\left\{\pi - \cot^{-1}\left(\frac{2}{3}\right)\right\}$

$y = -\tan\left(\cot^{-1}\frac{2}{3}\right)$

$\because$ $\cot^{-1}x = \tan^{-1}\frac{1}{x}$ if $x > 0$

$\therefore$ $y = -\tan\left(\tan^{-1}\frac{3}{2}\right)$ $\Rightarrow$ $y = -\frac{3}{2}$ so $\sin\left(\pi\tan\left\{\cot^{-1}\left(\frac{-2}{3}\right)\right\}\right) = \sin\left(-\frac{3\pi}{2}\right) = 1$

(ii) $\sin\left(2\tan^{-1}\frac{1}{2}\right) = 2\sin\left(\tan^{-1}\frac{1}{2}\right)\cos\left(\tan^{-1}\frac{1}{2}\right) = 2\sin\left(\sin^{-1}\frac{1}{\sqrt{5}}\right) \times \cos\left(\cos^{-1}\frac{2}{\sqrt{5}}\right)$

$= 2 \times \frac{1}{\sqrt{5}} \times \frac{2}{\sqrt{5}} = \frac{4}{5}$

(iii) $\cos\left(2\cos^{-1}\frac{1}{5} + \sin^{-1}\frac{1}{5}\right) = \cos\left(\cos^{-1}\frac{1}{5} + \sin^{-1}\frac{1}{5} + \cos^{-1}\frac{1}{5}\right)$

$= \cos\left(\frac{\pi}{2} + \cos^{-1}\frac{1}{5}\right) = -\sin\left(\cos^{-1}\left(\frac{1}{5}\right)\right)$ ....(i)

$= -\sqrt{1 - \left(\frac{1}{5}\right)^2} = -\frac{2\sqrt{6}}{5}$.

Self practices problem :

(23) Define (i) $\cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$ in terms of $\tan^{-1}x$

(ii) $\tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$ in terms of $\tan^{-1}x$

(24) Find the value of (i) $\sec\left(\cos^{-1}\left(\frac{2}{3}\right)\right)$, (ii) $\text{cosec}\left(\sin^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$,

(iii) $\tan\left(\text{cosec}^{-1}\frac{\sqrt{41}}{4}\right)$, (iv) $\sec\left(\cot^{-1}\frac{16}{63}\right)$, (v) $\sin\left\{\frac{1}{2}\cot^{-1}\left(\frac{-3}{4}\right)\right\}$

(vi) $\tan\left\{2\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right\}$,

(25) If $x \in (-1, 1)$ and $2\tan^{-1}x = \tan^{-1}y$ then find y in term of x.

(26) Find the value of $\sin(2\cos^{-1}x + \sin^{-1}x)$ when $x = \frac{1}{5}$

Answers : (23) (i) $\cos^{-1}\dfrac{1-x^2}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$

(ii) $\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right) = \begin{cases} 3\tan^{-1}x\ ; & -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ \pi + 3\tan^{-1}x\ ; & -\infty < x < -\dfrac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1}x\ ; & \dfrac{1}{\sqrt{3}} < x < \infty \end{cases}$

(24) (i) $\dfrac{3}{2}$ (ii) $-\sqrt{3}$ (iii) $\dfrac{4}{5}$ (iv) $\dfrac{65}{16}$ (v) $\dfrac{2\sqrt{5}}{5}$ (vi) $\dfrac{-7}{17}$

(25) $y = \dfrac{2x}{1-x^2}$

(26) $\dfrac{1}{5}$

Identities on addition and subtraction :

S.No.	Identities	Condition
(1)	$\tan^{-1}x + \tan^{-1}y = \pi/2$	$x, y > 0$ & $xy = 1$
(2)	$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$	$x, y \ge 0$ & $xy < 1$
(3)	$\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$	$x, y \ge 0$ & $xy > 1$
(4)	$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$	$x \ge 0, y \ge 0$
(5)	$\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$	$x \ge 0, y \ge 0$ and $(x^2 + y^2) \le 1$
(6)	$\sin^{-1}x + \sin^{-1}y = \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$	$x \ge 0, y \ge 0$ and $(x^2 + y^2) \ge 1$
(7)	$\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$	$x, y \in [0, 1]$
(8)	$\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$	$x, y \in [0, 1]$
(9)	$\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$	$0 \le x < y \le 1$
(10)	$\cos^{-1}x - \cos^{-1}y = -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$	$0 \le y < x \le 1$

Some useful Results :

(i) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, then $x + y + z = xyz$

(ii) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \dfrac{\pi}{2}$, then $xy + yz + zx = 1$

(iii) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$

(iv) $\tan^{-1}1 + \tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3} = \dfrac{\pi}{2}$

Example # 38 : Show that $\cos^{-1}\frac{4}{5} + \sin^{-1}\frac{15}{17} = \frac{\pi}{2} + \cos^{-1}\frac{84}{85}$

Solution : $\cos^{-1}\frac{4}{5} = \sin^{-1}\frac{3}{5}$

$\because \quad \frac{3}{5} > 0,\ \frac{15}{17} > 0$ and $\left(\frac{3}{5}\right)^2 + \left(\frac{15}{17}\right)^2 = \frac{8226}{7225} > 1$

$\therefore \quad \sin^{-1}\frac{3}{5} + \sin^{-1}\frac{15}{17} = \pi - \sin^{-1}\left(\frac{3}{5}\sqrt{1-\frac{225}{289}} + \frac{15}{17}\sqrt{1-\frac{9}{25}}\right)$

$= \pi - \sin^{-1}\left(\frac{3}{5}\ .\ \frac{8}{17} + \frac{15}{17}\ .\ \frac{4}{5}\right) = \pi - \sin^{-1}\left(\frac{84}{85}\right) = \pi - \frac{\pi}{2} + \cos^{-1}\frac{84}{85} = \frac{\pi}{2} + \cos^{-1}\frac{84}{85}$

Example # 39 : Evaluate $\cot^{-1}\frac{1}{9} + \cot^{-1}\frac{4}{5} + \cot^{-1}1$

Solution : $\cot^{-1}\frac{1}{9} + \cot^{-1}\frac{4}{5} + \cot^{-1}1 = \tan^{-1}9 + \tan^{-1}\frac{5}{4} + \cot^{-1}1$

$\because \quad 9 > 0,\ \frac{5}{4} > 0$ and $\left(9 \times \frac{5}{4}\right) > 1$

$\therefore \quad \tan^{-1}9 + \tan^{-1}\frac{5}{4} + \cot^{-1}1 = \pi + \tan^{-1}\left(\frac{9+\frac{5}{4}}{1-9.\frac{5}{4}}\right) + \cot^{-1}1 = \pi + \tan^{-1}(-1) + \cot^{-1}1$

$= \pi - \frac{\pi}{4} + \cot^{-1}1 = \pi$.

Self practice problems:

(27) Evaluate $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65}$

(28) If $\tan^{-1}4 + \tan^{-1}5 = \cot^{-1}\lambda$, then find 'λ'

(29) Prove that $2\cos^{-1}\frac{3}{\sqrt{13}} + \cot^{-1}\frac{16}{63} + \frac{1}{2}\cos^{-1}\frac{7}{25} = \pi$

Answers. (27) $\frac{\pi}{2}$ (28) $\lambda = -\frac{19}{9}$ (29) $x = \frac{1}{2}$